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ALMOST PERIODICITY FOR THE NONLINEAR CUBIC SCHRÖDINGER EQUATION

BY

LAURENT AMOUR*

Département de Mathématiques, URA CNRS 742
Institut Galilée, Université Paris XIII, 93430 Villetaneuse, France
e-mail: AMOUR@MOSCOU.UNIV-P13.FR
and
CMLA, ENS Cachan, URA CNRS 1611
61, avenue du président Wilson, 94235 Cachan Cedex, France
e-mail: AMOUR@CMLA.ENS-CACHAN.FR

ABSTRACT

For initial data in H^2 we prove that the periodic solutions in space to the nonlinear cubic Schrödinger equation are almost periodic in time.

1. Introduction

The nonlinear cubic Schrödinger equation

$$(1.1)\qquad \begin{aligned} &iu_t + u_{xx} - 2|u|^2u = 0, \\ &u(x+1,t) = u(x,t), \qquad\qquad (x,t) \in \mathbb{R}^2, \\ &u|_{t=0} = u_0, \quad u_0(x+1) = u_0(x), \end{aligned}$$

is well-posed in $L_1^4(\mathbb{R})$** [Bour1] and in $H_1^s(\mathbb{R})$ for all $s > 0$ [Bour2]. This second result of J. Bourgain motivates us to prove the almost periodicity of (1.1) for initial data u_0 in $H_1^2(\mathbb{R})$. In the following we prefer to rewrite (1.1) identifying

* *Current address:* Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel.

** The subscript 1 means that the elements of the space are 1-periodic.

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the complex $u(x,t)$ with $p(x,t)+iq(x,t)$ where p and q are real valued:

$$
(1.2)\qquad
\begin{gathered}
\begin{pmatrix} p_t \\ q_t \end{pmatrix} = T_2(p,q) = -\begin{pmatrix} p_{xx} \\ q_{xx} \end{pmatrix}^{\perp} + 2(p^2+q^2)\begin{pmatrix} p \\ q \end{pmatrix}^{\perp}, \\
p(x+1,t) = p(x,t), \quad q(x+1,t) = q(x,t), \\
(p,q)|_{t=0} = (p_0,q_0), \quad p_0(x+1) = p_0(x), \quad p_0(x+1) = p_0(x),
\end{gathered}
$$

for all $x \in \mathbb{R}$, where $\begin{pmatrix} a \\ b \end{pmatrix}^{\perp}$ denotes $\begin{pmatrix} b \\ -a \end{pmatrix}$.

The $AKNS$ system $\mathcal{H}(p,q)$ defined by

$$
(1.3)\qquad \mathcal{H}(p,q)F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx}F + \begin{pmatrix} -q(x) & p(x) \\ p(x) & q(x) \end{pmatrix} F
$$

is the self-adjoint operator of Lax pair associated with (1.1a).

Set $\mathcal{H}^j = \{(p,q) \in H^j_{\mathbb{R}}([0,1])^2 |\ p^{(k)}(1) = p^{(k)}(0),\ q^{(k)}(1) = q^{(k)}(0),\ k = 0,\dots,j-1\}$. Consider the operator $\mathcal{H}(p,q)$ with (p,q) in $H^1_1(\mathbb{R})^2$ and with the periodic self-adjoint boundary conditions $F(x+2) = F(x)$, and consider $\mathcal{H}(p,q)$ with (p,q) in $\mathcal{H}^1$ with the self-adjoint boundary conditions $F(1) = F(0)$ (or $F(1) = -F(0)$). These two operators have the same isospectral sets denoted $\mathrm{Iso}(p,q)$ by identifying $H^1_1(\mathbb{R})^2$ with $\mathcal{H}^1$. Moreover for $(p,q) \in \mathcal{H}^2$, $\mathrm{Iso}(p,q) \subset \mathcal{H}^2$. The spectrum of $\mathcal{H}(p,q)$ in $\mathcal{H}^1$ with the latter boundary conditions is a pure point spectrum. We denote by $(\lambda_k(p,q))_{k\in\mathbb{Z}}$ the increasing sequence of eigenvalues. Due to the Lax formulation each $\lambda_k(p,q)$ is a conserved quantity of (1.1). We then want to prove the almost periodicity in time of the flow $\big(p(t),q(t)\big)$ solution to $\begin{pmatrix} p_t \\ q_t \end{pmatrix} = T_2(p,q)$, $(p,q)|_{t=0} \in \mathcal{H}^2$. $\big(p(t),q(t)\big) \in \mathrm{Iso}(p_0,q_0) \subset \mathcal{H}^2$ is the flow associated with the vector field $(p,q) \mapsto T_2(p,q)$ tangent to $\mathrm{Iso}(p_0,q_0)$.

In 1974–75 the periodicity in time for the KdV equation has been studied by P. Lax [Lax1] [Lax2], V. A. Marchenko [Mar], S. P. Novikov and B. A. Dubrovin [Nov], H. P. McKean and P. Van Moeberke [McK-VanMoe]. In 1976 H. P. McKean and E. Trubowitz [McK-Tru] gave the result of the almost periodicity in H^3_1 for KdV. Recently J. Bourgain [Bour3] has shown how to extend the works of [McK-Tru] in L^2_1.

For solving the case of the nonlinear Schrödinger equation in H^2_1 we used the works of [McK-Tru] and the results of [Gré-Gui].

We now give briefly some results related to the periodic $AKNS$ operator. These results have been established in [Gré] and [Gré-Gui]. See these references for precise details.

We denote by $F_1(x,\lambda,p,q)$ [resp. $F_2(x,\lambda,p,q)$] the solution to $\mathcal{H}(p,q)F=\lambda F$ with $F_1(0,\lambda,p,q)=(1,0)^T$ [resp. $F_2(0,\lambda,p,q)=(0,1)^T$]. For $i=1$ and 2, Y_i and Z_i are the two components of F_i. The eigenvalues λ_k's are the zeros of the entire function $\lambda\mapsto\Delta^2(\lambda)-4=\big(Y_1(1,\lambda)+Z_2(1,\lambda)\big)^2-4$. Set $m_\pm(\lambda)=\frac{1}{2}\big[\Delta(\lambda)\pm(\Delta^2(\lambda)-4)^{\frac{1}{2}}\big]$, $b_\pm=\big((m_\pm(\lambda)-Y_1(1,\lambda)\big)/Y_2(1,\lambda)=Z_1(1,\lambda)/\big(m_\pm(\lambda)-Z_2(1,\lambda)\big)$ and $f_\pm(x,\lambda)=F_1(x,\lambda)+b_\pm(\lambda)F_2(x,\lambda)$. Then $f_+(x,\lambda_k)=f_-(x,\lambda_k)$ is an eigenfunction associated with the eigenvalue λ_k.

For $a=\begin{pmatrix}a_1\\a_2\end{pmatrix}$ and $b=\begin{pmatrix}b_1\\b_2\end{pmatrix}$ two elements of $\mathbb{C}^2$, we define the following bilinear forms:

$$u(a,b)=a_1b_1-a_2b_2,\quad v(a,b)=a_1b_2+a_2b_1,\quad w(a,b)=a_1b_1+a_2b_2.$$

We denote by $\nabla_{p,q}G$ the vector $\begin{pmatrix}\partial G/\partial p\\ \partial G/\partial q\end{pmatrix}$ where $\partial G/\partial f$ is the Frêchet derivative with respect to f of G. For $(p,q)\in L^2_{\mathbb{R}}([0,1])^2$,

$$\text{(1.4)}\quad \nabla_{p,q}\lambda_k=-\frac{\nabla_{p,q}\Delta(\lambda_k)}{\dfrac{\partial\Delta}{\partial\lambda}(\lambda_k)},\quad \nabla_{p(x),q(x)}\Delta(\lambda)=\begin{pmatrix}v(f_+(x,\lambda),f_-(x,\lambda)\\ -u(f_+(x,\lambda),f_-(x,\lambda))\end{pmatrix}.$$

The Dirichlet spectrum is the spectrum of $\mathcal{H}(p,q)$ associated with the boundary conditions $Y(0)=Y(1)=0$. The Dirichlet spectrum is a strictly increasing sequence of eigenvalues of multiplicity one and denoted $\big(\mu_k(p,q)\big)_{k\in\mathbb{Z}}$.

Finally we recall that the map $K\times\mu$ defined by

$$(K\times\mu)(p,q)=\Big(\big(Z_2(1,\mu_j(p,q),p,q)-(-1)^j\big)_{j\in\mathbb{Z}},\quad\big(\mu_j(p,q)-j\pi\big)_{j\in\mathbb{Z}}\Big)$$

is a real global bianalytic coordinate system on $L^2_{\mathbb{R}}([0,1])^2$ into $\ell^2_{\mathbb{R}}(\mathbb{Z})^2$. This coordinate system allows a parameterization of the isospectral sets $\operatorname{Iso}(p,q)$, and generically $\operatorname{Iso}(p,q)$ is an infinite product of circles.

The following is divided into two parts. In the first section we give a sequence of tangent vectors to $\operatorname{Iso}(p,q)$ defined by induction, which leads to the expansion of T_2 on the family of tangent vectors $\nabla^{\perp}_{p,q}\Delta(\lambda_{2k})$'s. In the second section the similar expansions of each element of another sequence of tangent vectors generating periodic flows in time, leads to the almost periodicity of the flow associated with T_2.

Finally note the recent results of [BBGK] related to Theorem 3.2.

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2. A class of tangent vectors

The vectors

$$T_0 = \begin{pmatrix} p \\ q \end{pmatrix}^{\perp},$$

$$T_1 = \begin{pmatrix} p_x \\ q_x \end{pmatrix},$$

$$T_2 = -\begin{pmatrix} p_{xx} \\ q_{xx} \end{pmatrix}^{\perp} + 2(p^2+q^2)\begin{pmatrix} p \\ q \end{pmatrix}^{\perp}$$

are tangent to $\mathrm{Iso}(p,q)$ and the solutions to $(p_t,q_t) = T_i(p,q)$ with initial data (p_0,q_0) $(i=0,1,2)$ are isospectral, i.e. for all t, the periodic spectrum of the operator $\mathcal{H}(p(t),q(t))$ is the same as that of the operator $\mathcal{H}(p_0,q_0)$. The rotation flow $(p(t),q(t)) = (p_0\cos t + q_0 \sin t, -p_0 \sin t + q_0 \cos t)$ is associated with T_0 and it is isospectral (cf. [Am, Th. 2.1]). The translation flow $(p(x,t),q(x,t)) = (p_0(x-t), q_0(x-t))$ corresponds with T_1; it is obviously isospectral. The flow associated with T_2 is a solution to the nonlinear cubic Schrödinger equation. These vectors are in fact the three first of a sequence defined by induction of tangent vectors to $\mathrm{Iso}(p,q)$.

For g a sufficiently smooth function of x we denote by $\mathcal{P}[g]$ one primitive with respect to x of g.

THEOREM 2.1: *For $(p,q) \in C_1^\infty([0,1])^2$ each vector of the sequence*

$$T_0 = \begin{pmatrix} p \\ q \end{pmatrix}^{\perp},$$

$$T_{m+1} = -D_x T_m^{\perp} + 4\mathcal{P}[w(\begin{pmatrix} p \\ q \end{pmatrix}, T_m)]\begin{pmatrix} p \\ q \end{pmatrix}^{\perp}$$

is tangent to $\mathrm{Iso}(p,q)$.

Remark 1: To obtain the complex form of Theorem 2.1 one identifies $L^2_{\mathbb{C}}([0,1])$ with $L^2_{\mathbb{R}}([0,1])^2$ by $u = p + iq$. This gives the following sequence of tangent vectors:

$$T_0 = -iu, \quad T_{m+1} = iD_xT_m - 2iu\mathcal{P}[u\bar{T}_m + \bar{u}T_m].$$

In particular a special choice of $\mathcal{P}$ gives $T_1 = u_x$ and $T_2 = iu_{xx} - 2i|u|^2u$. The first and second primitives take the values 0 and $|u(0)|^2$ respectively, at $x = 0$.

Remark 2: In Theorem 2.1 the fact that $\mathcal{P}$ is defined up to an additive constant is possible since it consequently adds for each T_m a constant multiplied by the tangent vector $\begin{pmatrix} p \\ q \end{pmatrix}^{\perp} = T_0$.

We will use the following lemma in the proof of Theorem 2.1. The wronskian of F and G is defined by $[F, G] = w(F, G^{\perp})$.

LEMMA 2.2: *For $(p, q) \in L^2_{\mathbb{R}}([0,1])^2$, suppose that $\mathcal{H}(p,q)F = \mu F$ and $\mathcal{H}(p,q)G = \nu G$. Then*

$$D_x \begin{pmatrix} u(F,G) \\ v(F,G) \end{pmatrix} + 2 \begin{pmatrix} p \\ q \end{pmatrix} w(F,G) = (\mu + \nu) \begin{pmatrix} v(F,G) \\ -u(F,G) \end{pmatrix}, \tag{2.1}$$

$$D_x w(F,G) + 2p\, u(F,G) + 2q\, v(F,G) = (\nu - \mu)[F,G]. \tag{2.2}$$

Proof of Lemma 2.2: Suppose $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ satisfy the hypothesis of the Lemma. Multiply the two lines of the system $\mathcal{H}(p,q)F = \mu F$ by G_2 and G_1 respectively and the two equations. Add to this equation the similar one obtained by exchanging F and G as well as μ and ν to obtain the first equation of (2.1). The second equation of (2.1) and (2.2) can be proved in a similar way. ∎

Proof of Theorem 2.1: For the solution to $(p_t, q_t) = T_m(p,q)$ we define for all $j \in \mathbb{Z}$, $T_m\lambda_j = \frac{d}{dt}\lambda_j(p,q)$. Using (1.4) and Lemma 2.2 with $(F, G, \mu, \nu) = (f, f, \lambda_j, \lambda_j)$ where $f(x) = f_+(x, \lambda_j) = f_-(x, \lambda_j)$, we have

$$\begin{aligned} 2\lambda_j T_m \lambda_j &= \langle 2\lambda_j \nabla_{p,q}\lambda_j; T_m \rangle \\ &= \left\langle D_x \begin{pmatrix} u(f,f) \\ v(f,f) \end{pmatrix} + 2 \begin{pmatrix} p \\ q \end{pmatrix} w(f,f);\ T_m \right\rangle. \end{aligned}$$

Integrating by parts the preceding equation and using (2.2) we obtain

$$\begin{aligned} 2\lambda_j T_m \lambda_j &= \langle 2\lambda_j \nabla_{p,q}\lambda_j; T_m \rangle \\ &= \left\langle \begin{pmatrix} u(f,f) \\ v(f,f) \end{pmatrix};\ D_x T_m - 4\mathcal{P}[w(\begin{pmatrix} p \\ q \end{pmatrix}, T_m)] \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle. \end{aligned}$$

Thus

$$2\lambda_j T_m \lambda_j = T_{m+1}\lambda_j, \quad \forall j \in \mathbb{Z}.$$

Consequently T_m is tangent for all $m \geq 1$ if T_0 is tangent. ∎

Remark 3: The fact that $(p,q) \in C^\infty_{\mathbb{R}}([0,1])^2$ is used for the integration by parts in the preceding proof for all $m \in \mathbb{N}$. Of course for $m \le 2$ it is sufficient to consider $(p,q) \in \mathcal{H}^2$.

The interpolation theorem below will be used in the following. Define

$$E = \Big\{ F(z) \text{ entire of order 1 and with type at most 1, such that } \|F\|_E = \int_{-\infty}^{+\infty} |F(z)|^2 dz < +\infty \Big\}.$$

The interpolation points are the μ_i, $i \in \mathbb{Z}$ sequence of Dirichlet eigenvalues for an operator $\mathcal{H}(p,q)$ and zero of $\lambda \mapsto Y_2(1,\lambda,p,q)$.

THEOREM 2.3 (Theorem of interpolation): *The restriction application*

$$F \mapsto \big(F(\mu_i)\big)_{i\in\mathbb{Z}}$$

is one to one from E into $\ell^2_{\mathbb{C}}(\mathbb{Z})$. Moreover, the converse application is given by the formula

$$F(z) = \sum_{i\in\mathbb{Z}} F(\mu_i) \frac{Y_2(1,z)}{\frac{\partial Y_2}{\partial z}(1,\mu_i)(z-\mu_i)} \quad , \quad \forall z \in \mathbb{C}.$$

Proof of Theorem 2.3: It is an adaptation of [McK-Tru, sec. 5]. See this reference for precise details. We fix $(p,q) \in L^2_{\mathbb{R}}([0,1])$ in the following.

Set $e(z) = Y_2(1,z) + iZ_2(1,z)$. We have $|e(z)| > |e(z^*)|$ for $\operatorname{Im} z > 0$.

Indeed the development $e(z)\big(e(z)\big)^* - e(z^*)\big(e(z^*)\big)^*$ gives

$$\begin{aligned} |e(z)|^2 - |e(z^*)|^2 &= 2i\big(Y_2^*(1,z)Z_2(1,z) - Y_2(1,z)Z_2^*(1,z)\big) \\ &= 4\operatorname{Im} \int_0^1 D_x\big(Y_2(x,z)Z_2^*(x,z)\big)dx. \end{aligned}$$

By direct calculus

$$\begin{aligned} D_x\big(Y_2(x,z)Z_2^*(x,z)\big) = &- q(x)\big(|Y_2(x,z)|^2 + |Z_2(x,z)|^2\big) \\ &+ z|Z_2(x,z)|^2 - z^*|Y_2(x,z)|^2. \end{aligned}$$

We then obtain $|e(z)|^2 - |e(z^*)|^2 > 0$ since $q(x)$ is real and $-\operatorname{Im} z^* = \operatorname{Im} z > 0$.

We define B the subset of entire functions as in [McK-Tru, p. 169] or [De Bran, p. 52] with the scalar product $B[F,G] = \frac{1}{\pi}\int_{-\infty}^{+\infty} F(z)G^*(z)|e(z)|^{-2}\,dz$, and A the set of functions F defined on the μ_i with $A[F,F] = \sum N_i^{-2}|F(\mu_i)|^2 < +\infty$, where

$N_i^2 = \|F(\cdot, \mu_i)\|^2_{L^2_{\mathbb{R}}([0,1])^2} = \frac{\partial Y_2}{\partial z}(1, \mu_i) Z_2(1, \mu_i)$. A and B are Hilbert space. The functions

$$z \mapsto 1_\alpha(z) = \big(e^*(\alpha)e(z) - e(\alpha^*)e^*(z^*)\big)/2i(\alpha^* - z) \tag{2.3}$$

generate A and B. Moreover, for all $F \in B$ we have $F(z) = B[F, 1_z]$.

The key of the proof is then to prove

$$A[1_\alpha, 1_\beta] = B[1_\alpha, 1_\beta], \quad \forall(\alpha, \beta) \in \mathbb{C}^2, \tag{2.4}$$

which shows that the restriction application is an isomorphism from B to A. Then we can identify B with E and A with $\ell^2_{\mathbb{C}}(\mathbb{Z})$ (cf [McK-Tru, p. 172]). Let us prove (2.4). With (2.3) we have

$$1_\alpha(\mu_k) = \frac{Z_2(1, \mu_k) Y_2^*(\alpha)}{\alpha^* - \mu_k}, \quad \forall k \in \mathbb{Z} \tag{2.5}$$

hence

$$\begin{aligned} A[1_\alpha, 1_\beta] &= \sum_{k \in \mathbb{Z}} N_k^{-2}\, 1_\alpha(\mu_k) 1_\alpha^*(\mu_k) \\ &= Y_2^*(1, \alpha) Y_2(1, \beta) \sum_{k \in \mathbb{Z}} N_k^{-2} \frac{Z_2^2(1, \mu_k)}{(\alpha^* - \mu_k)(\beta - \mu_k)} \\ &= \frac{Y_2^*(1, \alpha) Y_2(1, \beta)}{\beta - \alpha^*} \sum_{k \in \mathbb{Z}} N_k^{-2} \left(\frac{Z_2^2(1, \mu_k)}{\alpha^* - \mu_k} - \frac{Z_2^2(1, \mu_k)}{\beta - \mu_k} \right). \end{aligned} \tag{2.6}$$

The two sums in (2.6) are expressed using the given expansions in [Lev-Sar] of the kernel of the resolvant $G(x, y, \lambda)$ of the operator $\mathcal{H}(p, q) - \lambda$. $G(x, y, \lambda)$ is given by [Lev-Sar, p. 195] for $x \neq y$ and [Lev-Sar, Lemma 11.3.2, p. 314] shows that the application $(x, y) \mapsto G_{22}(x, y, \lambda)$ is continuous through the line $x = y$. We then obtain with the definition of $\omega(\lambda), \varphi(x, \lambda)$ and $\varphi_n(x, \lambda)$ of [Lev-Sar, (4.6), p. 319]:

$$\frac{\varphi_2(1, \lambda)}{\omega(\lambda)} = G_{22}(1, 1, \lambda) = \sum_{n \in \mathbb{Z}} \frac{\left[\varphi_n^T(1, \lambda) \varphi_n(1, \lambda)\right]_{22}}{\lambda - \mu_n}. \tag{2.7}$$

Observe that λ has been changed in $-\lambda$ to pass from the definition of the operator L in [Lev-Sar, (3.2), p. 310] to the definition of $\mathcal{H}(p, q)$. Clearly $\varphi_n(x) = F_2(x, \mu_n)\big(\dot{Y}_2(1, \mu_n) Z_2(1, \mu_n)\big)^{-\frac{1}{2}}$, $\omega(\lambda) = \varphi_1(1, \lambda) = Y_2(1, \lambda)$ and from (2.7)

$$\frac{Z_2(1, \lambda)}{Y_2(1, \lambda)} = \sum_{n \in \mathbb{Z}} (\lambda - \mu_n)^{-1} \frac{Z_2^2(1, \mu_n)}{N_n^2}$$

which is used for the computation of (2.6). We obtain

$$A[1_\alpha, 1_\beta] = (\beta - \alpha^*)^{-1}\big(Z_2^*(1,\alpha)Y_2(1,\beta) - Y_2^*(1,\alpha)Z_2(1,\beta)\big). \tag{2.8}$$

Besides, a direct calculus shows $1_\alpha(\beta)$ is equal to the right hand side of (2.8). Equality (2.4) is then verified since $1_\alpha(\beta) = B[1_\alpha, 1_\beta]$.

The interpolation formula comes immediately from $F(z) = A[F, 1_z]$ for $F \in E$ using (2.5). ∎

We know that for all λ, $\nabla^\perp_{p,q}\Delta(\lambda)$ is tangent to $\mathrm{Iso}(p,q)$. In particular for all $j \in \mathbb{Z}$, $\nabla^\perp_{p,q}\Delta(\lambda_{2j})$ is tangent to $\mathrm{Iso}(p,q)$. Using the interpolation theorem we will give an expansion in $\nabla^\perp_{p,q}\Delta(\lambda_{2j})$ of each T_m. Before, we must check that each component of $\nabla_{p,q}\Delta(\lambda)$ is an entire function of λ which belongs to E, and consequently check it is in $L^2(\mathbb{R})$. This results from the following theorem together with the asymptotic behavior of F_1 and F_2 for (p,q) in $\mathcal{H}^2$.

$(T_t g)(x)$ denotes $g(t+x)$ for any g function of x.

THEOREM 2.4: *For* $(p,q) \in L^2_{\mathbb{R}}([0,1])^2$

$$\nabla_{p(t),q(t)}\Delta(\lambda) = -F_1(1,\lambda,T_tp,T_tq) + F_2^\perp(1,\lambda,T_tp,T_tq).$$

Proof of Theorem 2.4: It is easy to see that

$$\begin{aligned} F_1(x,\lambda,T_tp,T_tq) =& Z_2(t,\lambda,p,q)F_1(x+t,\lambda,p,q) \\ &- Z_1(t,\lambda,p,q)F_2(x+t,\lambda,p,q), \\ F_2(x,\lambda,T_tp,T_tq) =& - Y_2(t,\lambda,p,q)F_1(x+t,\lambda,p,q) \\ &+ Y_1(t,\lambda,p,q)F_2(x+t,\lambda,p,q), \end{aligned} \tag{2.9}$$

since the left hand side and the right hand side of (2.9a) [resp. (2.9b)] verify the same system with the same data at $x = 0$. Besides, we have from [Gré, Th. 2(i)(ii), p. 132]

$$\begin{aligned} \nabla_{p,q}\Delta(\lambda,p,q) = & \begin{pmatrix} v\big(Y_2(1)F_1(t) + Z_2(1)F_2(t), F_1(t)\big) \\ -u\big(Y_2(1)F_1(t) + Z_2(1)F_2(t), F_1(t)\big) \end{pmatrix} \\ & - \begin{pmatrix} v\big(Y_1(1)F_1(t) + Z_1(1)F_2(t), F_1(t)\big) \\ -u\big(Y_1(1)F_1(t) + Z_1(1)F_2(t), F_1(t)\big) \end{pmatrix} \end{aligned}$$

where the right members are evaluated at (λ,p,q). Consequently

$$\nabla_{p,q}\Delta(\lambda,p,q) = \begin{pmatrix} v\big(F_2(1+t), F_1(t)\big) - v\big(F_1(1+t), F_2(t)\big) \\ -u\big(F_2(1+t), F_1(t)\big) + u\big(F_1(1+t), F_2(t)\big) \end{pmatrix}.$$

Considering (2.8) and (2.9) for $x = 1$ it becomes

$$v\big(F_2(1+t), F_1(t)\big) - v\big(F_1(1+t), F_2(t)\big) = -Y_1(1, \lambda, T_t p, T_t q) + Z_2(1, \lambda, T_t p, T_t q)$$

and

$$-u\big(F_2(1+t), F_1(t)\big) + u\big(F_1(1+t), F_2(t)\big) = -Y_2(1, \lambda, T_t p, T_t q) - Z_1(1, \lambda, T_t p, T_t q)$$

which proves the theorem. ∎

The asymptotic behaviors of $F_1(x, \lambda, p, q)$ and $F_2(x, \lambda, p, q)$ for $(p, q) \in \mathcal{H}^2$ are given by [Gré, (2.1-2), p. 121]. We then deduce with Theorem 2.4

$$\nabla_{p(t),q(t)} \Delta(\lambda) = \frac{2\sin\lambda}{\lambda} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} + O(\lambda^{-2}) \tag{2.10}$$

for $(p, q) \in \mathcal{H}^2$ with $O(\lambda^{-2})$ uniformly in t.

For (p, q) given there exist $(p_1, q_1) \in \operatorname{Iso}(p, q)$ with $\mu_i(p_1, q_1) = \lambda_{2i}(p, q)$, $\forall i \in \mathbb{Z}$. (This is a consequence of [Gré, Th. 7, p. 138].) The chosen interpolation points are the $\lambda_{2i}(p, q) = \mu_i(p_1, q_1)$. Thus

$$\nabla_{p,q} \Delta(\lambda, p, q) = \sum_{i \in \mathbb{Z}} \nabla_{p,q} \Delta(\lambda_{2i}, p, q) \frac{Y_2(1, \lambda, p_1, q_1)}{\dot{Y}_2(1, \lambda_{2i}, p_1, q_1)(\lambda - \lambda_{2i})} \tag{2.11}$$

($\dot{}$ denotes $\partial/\partial\lambda$). We prefer to consider the normalized eigenfunctions

$$f_k(x, p, q) = f_\pm(x, \lambda_k(p, q), p, q) \| f_\pm(\cdot, \lambda_k(p, q), p, q) \|^{-1}_{L^2_{\mathbb{R}}([0,1])^2}.$$

LEMMA 2.5: *For $(p, q) \in L_{\mathbb{R}}([0, 1])^2$*

$$\nabla_{p(x),q(x)} \Delta(\lambda_{2i}) = -\dot{\Delta}(\lambda_{2i}, p, q) \begin{pmatrix} v\big(f_{2i}(x), f_{2i}(x)\big) \\ -u\big(f_{2i}(x), f_{2i}(x)\big) \end{pmatrix}.$$

Proof of Lemma 2.5: From (1.4) we have for all $i \in \mathbb{Z}$,

$$\nabla_{p,q} \Delta(\lambda_{2i}) = c_i \begin{pmatrix} v(f_{2i}, f_{2i}) \\ -u(f_{2i}, f_{2i}) \end{pmatrix}.$$

It remains to evaluate c_i. Obviously

$$w\big(f_+(x, \lambda), f_-(x, \lambda)\big)\big|_{\lambda = \lambda_{2i}} = c_i w\big(f_{2i}(x), f_{2i}(x)\big).$$

Thus

$$c_i = \int_0^1 w\big(f_+(x,\lambda), f_-(x,\lambda)\big)|_{\lambda=\lambda_{2i}}\,dx. \tag{2.12}$$

The same calculus as in [Gré, p. 132–133] shows that for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} Y_2(1)w\big(f_+(x), f_-(x)\big) =& Y_2(1)w\big(F_1(x), F_1(x)\big) - Z_1(1)w\big(F_2(x), F_2(x)\big) \\ &+ \big(Z_2(1) - Y_1(1)\big)w\big(F_1(x), F_2(x)\big). \end{aligned} \tag{2.13}$$

If $F = (Y, Z)^T$ is a solution to $\mathcal{H}(p,q)F = \lambda F$, then

$$-\dot{Z}_x - q\dot{Y} + p\dot{Z} = Y + \lambda\dot{Y}, \tag{2.14a}$$

$$\dot{Y}_x + p\dot{Y} + q\dot{Z} = Z + \lambda\dot{Z}, \tag{2.14b}$$

where the subscript x denotes the derivation with respect to x. If $\tilde{F} = (\tilde{Y}, \tilde{Z})^T$ is also a solution to $\mathcal{H}(p,q)\tilde{F} = \lambda\tilde{F}$, then (2.14a) multiplied by $\tilde{Y}$ and added to (2.14b) multiplied by $\tilde{Z}$ gives

$$w(F, \tilde{F}) = \dot{Y}\tilde{Z} - \dot{Z}\tilde{Y}. \tag{2.15}$$

Integrating (2.15), knowing that for $i = 1, 2$, $\dot{Y}_i(0,\lambda) = \dot{Z}_i(0,\lambda) = 0$ for all $\lambda \in \mathbb{C}$, we obtain

$$\begin{aligned} \int_0^1 w\big(F_1(x,\lambda), F_2(x,\lambda)\big)\,dx =& \dot{Y}_1 Z_2 - \dot{Z}_1 Y_2|_{(x=1,\lambda)}, \\ \int_0^1 w\big(F_1(x,\lambda), F_1(x,\lambda)\big)\,dx =& \dot{Y}_1 Z_1 - \dot{Z}_1 Y_1|_{(x=1,\lambda)}, \\ \int_0^1 w\big(F_2(x,\lambda), F_2(x,\lambda)\big)\,dx =& \dot{Y}_2 Z_2 - \dot{Z}_2 Y_2|_{(x=1,\lambda)}. \end{aligned} \tag{2.16}$$

From the computation of $\int_0^1 dx$ of the right hand side of (2.13) using expressions (2.16) for $\lambda = \lambda_{2i}$ and using $[F_1(x,\lambda), F_2(x,\lambda)] = 1$, we have

$$\int_0^1 w\big(f_+(x,\lambda), f_-(x,\lambda)\big)|_{\lambda=\lambda_{2i}}\,dx = -\dot{Y}_1(1,\lambda_{2i}) - \dot{Z}_2(1,\lambda_{2i})$$

which ends the proof of Lemma 2.5. ∎

Equality (2.11) becomes with Lemma 2.5:

$$\nabla_{p(x),q(x)}\Delta(\lambda,p,q) = \sum_{i\in\mathbb{Z}} \frac{Y_2(1,\lambda,p_1,q_1)}{\dot{Y}_2(1,\lambda_{2i},p_1,q_1)(\lambda-\lambda_{2i})}$$

$$\times -\dot{\Delta}(\lambda_{2i},p,q)\begin{pmatrix} v\big(f_{2i}(x,p,q),f_{2i}(x,p,q)\big) \\ -u\big(f_{2i}(x,p,q),f_{2i}(x,p,q)\big)\end{pmatrix} \tag{2.17}$$

$$= -\sum_{i\in\mathbb{Z}} \varepsilon_i \frac{Y_2(1,\lambda,p_1,q_1)}{\lambda-\lambda_{2i}\ X_i^{\perp}} \tag{2.18}$$

where

$$X_i = \begin{pmatrix} u\big(f_{2i}(x,p,q),f_{2i}(x,p,q)\big) \\ v\big(f_{2i}(x,p,q),f_{2i}(x,p,q)\big)\end{pmatrix} \quad \text{and} \quad \varepsilon_i = \frac{\dot{\Delta}(\lambda_{2i},p,q)}{\dot{Y}_2(1,\lambda_{2i},p_1,q_1)}.$$

The method used in [McK-Tru, Th. 6.1] shows that $\varepsilon_i = O(\lambda_{2i}-\lambda_{2i-1})$. We deduce from (2.10) and (2.18) that for $|\lambda|$ large

$$\begin{pmatrix} 2p \\ 2q \end{pmatrix} \sim \frac{\lambda}{Y_2(1,\lambda)}\nabla_{p,q}\Delta(\lambda) = -\sum_{i\in\mathbb{Z}} \frac{\lambda}{\lambda-\lambda_{2i}}\varepsilon_i \begin{pmatrix} v(f_{2i},f_{2i}) \\ -u(f_{2i},f_{2i})\end{pmatrix},$$

that is to say

$$T_0 = \begin{pmatrix} p \\ q \end{pmatrix}^{\perp} = \frac{1}{2}\sum_{i\in\mathbb{Z}} \varepsilon_i X_i. \tag{2.19}$$

Applying to (2.19) the operator $\mathcal{L}$ defined by

$$\mathcal{L}T = -D_x T^{\perp} + 4\Big(\int_0^x w\Big(\begin{pmatrix} p \\ q \end{pmatrix},T\Big)\,ds\Big)\begin{pmatrix} p \\ q \end{pmatrix}^{\perp},$$

we obtain using (2.1–2) with $\nu=\mu=\lambda$ and $F=G=f_{2i}$

$$\begin{aligned}\begin{pmatrix} p_x \\ q_x \end{pmatrix} = T_1 &= \sum_{i\in\mathbb{Z}} \varepsilon_i\lambda_{2i}\begin{pmatrix} u(f_{2i},f_{2i}) \\ v(f_{2i},f_{2i})\end{pmatrix} + 2c\begin{pmatrix} p \\ q \end{pmatrix}^{\perp} \\ &= \sum_{i\in\mathbb{Z}} \varepsilon_i(\lambda_{2i}+c)\begin{pmatrix} u(f_{2i},f_{2i}) \\ v(f_{2i},f_{2i})\end{pmatrix}\end{aligned} \tag{2.20}$$

with

$$2c = \sum_{i\in\mathbb{Z}} \varepsilon_i w(f_{2i},f_{2i})|_{x=0}. \tag{2.21}$$

In the same way $\mathcal{L}$ applied to T_1 gives

$$T_2 = \sum_{i\in\mathbb{Z}} \varepsilon_i\big(2\lambda_{2i}(\lambda_{2i}+c)+d\big)\begin{pmatrix} u(f_{2i},f_{2i}) \\ v(f_{2i},f_{2i}) \end{pmatrix} \tag{2.22}$$

with

$$d = \Big(\sum_{i\in\mathbb{Z}} \varepsilon_i(\lambda_{2i}+c)w(f_{2i},f_{2i})|_{x=0}\Big) + p^2(0)+q^2(0). \tag{2.23}$$

Remark 4: We can repeat this process to obtain the expansions in X_i of each T_m for $m \geq 0$.

LEMMA 2.6: *The real valued functions c and d are constant on* Iso(p,q).

Proof of Lemma 2.6: Add the first equation of (2.19) multiplied by p to the second equation of (2.19) multiplied by q and use (2.2) to obtain $0 = \sum \varepsilon_i D_x w(f_{2i},f_{2i})$ for all $x \in [0,1]$. We then apply $\int_0^1 dx \int_0^x ds$ to the preceding equation and obtain

$$\sum \varepsilon_i w(f_{2i},f_{2i})|_{x=0} = \sum \varepsilon_i,$$

which concludes the proof for c with (2.21). In the same way we deduce from (2.20)

$$\|(p,q)\|^2_{L^2_{\mathbb{R}}([0,1])^2} = \Big(\sum_{i\in\mathbb{Z}} \varepsilon_i(\lambda_{2i}+c)\big(w(f_{2i},f_{2i})|_{x=0}-1\big)\Big) + p^2(0)+q^2(0),$$

and consequently, using (2.23), d is constant on Iso(p,q) since $\|(p,q)\|_{L^2_{\mathbb{R}}([0,1])}$ is itself a spectral invariant of the periodic problem. ∎

3. Almost periodicity in time

We recall that for each $k \in \mathbb{Z}$ the flow associated with the tangent vector field $V_k = \nabla^{\perp}_{p,q}\Delta(\mu_k)$ exists for all time and along this flow, only the kth coordinate (K_k,μ_k) of the coordinate system $K \times \mu$ is not stationary. More precisely this coordinate is never stationary. Since (K_k,μ_k) describes completely a circle in finite time, the flow $(p_k(\cdot,t),q_k(\cdot,t))$ associated to V_k is periodic. To evaluate the period π_k of the flow $(p_k(t),q_k(t))$ we consider without loss of generality that the initial data (p_1,q_1) at $t=0$ verifies $\mu_i(p_1,q_1) = \lambda_{2i-1}(p_1,q_1), \forall i \in \mathbb{Z}$. The evolution of μ_k along the flow $(p_k(t),q_k(t))$ is given by the differential equation

$\partial_t \mu_k = \pm\sqrt{\Delta^2(\mu_k) - 4}$. The proof and details about the sign $\pm$ are given in [Gré, pp. 136–137].

We then define $\pi_k/2$ as the smaller positive real such that $\mu_k(p_k(\pi_k), q_k(\pi_k)) = \lambda_{2k}$. Since $\pi_k/2 = \int_0^{\pi_k/2} \mathrm{dt}$, by the following change of variable $\mu = \mu_k(p_k(t), q_k(t))$ we obtain

$$\frac{\pi_k}{2} = \int_{\lambda_{2k-1}}^{\lambda_{2k}} \frac{d\mu}{\sqrt{\Delta^2(\mu) - 4}}. \tag{3.1}$$

We then prove easily (cf. [McK-Tru, Lemma 11.1]) that π_k is comparable to 1, i.e. π_k is minorazed and majorazed by two real strictly positive numbers, independent of k. Thus the associated flow to $W^k = \pi_k V_k$ is 1-periodic in time.

We define $\tau = (\tau_j)_{j\in\mathbb{Z}}$ and the sequences $\omega^i = (\omega^i_j)_{j\in\mathbb{Z}}$ for each $i \in \mathbb{Z}$ by

$$\tau_j = \varepsilon_i\big(d + 2\lambda_{2j}(c + \lambda_{2j})\big) \quad \text{and} \quad \omega^i_j = \pi_i \frac{Y_2(1,\mu_i)}{\dot{Y}_2(1,\lambda_{2j})(\mu_i - \lambda_{2j})}, \tag{3.2}$$

so that for each $i \in \mathbb{Z}$, $W^i = \sum_{j\in\mathbb{Z}} \omega^i_j X_j$ and $T_2 = \sum_{j\in\mathbb{Z}} \tau_j X_j$ (cf. (2.17)).

We denote by I the set of real sequences x indexed on $\mathbb{Z}$ such that $\|x\|_I = \sum x_i^2 (1+i^2)^{-3} < +\infty$. Clearly $\tau \in I$ and the sequences $\omega^i \in I$ for each $i \in \mathbb{Z}$.

LEMMA 3.1:

(i) *Every $x \in I$ can be written as $\sum_{i\in\mathbb{Z}} y_i\omega^i$ with $y \in I$.*

(ii) *For $L = \{x = \sum y_i\omega^i \in I$ with $y \in I$ and y_i integer for all $i\}$ the quotient space I/L is compact.*

Proof of Lemma 3.1: (i) If $\alpha \in I$ is orthogonal to ω^i for all $i \in \mathbb{Z}$, then

$$0 = \sum_{j\in\mathbb{Z}} (1+j^2)^{-3}\alpha_j\omega^i_j = \pi_i F(\mu_i), \quad \forall i \in \mathbb{Z}$$

with $F \in E$ and $F(\lambda_{2j}) = (1+j^2)^{-3}\alpha_j$ (F does not depend on i) using formula (3.2) for ω^i_j and the interpolation theorem at the points λ_{2j}. Thus $F \equiv 0$ using again the interpolation theorem at the points μ_j and then $\alpha_j = 0$ for all $j \in \mathbb{Z}$.

For all $x \in I$ we define $\hat{x}$ by $\hat{x}_j = (1+j^2)^{-3/2}x_j$ for every $j \in \mathbb{Z}$. Thus $x \in I \Leftrightarrow \hat{x} \in \ell^2_{\mathbb{R}}(\mathbb{Z})$. We then consider $\hat{x}$ as an element of the dual of E by $\hat{x}(F) = \sum F(\lambda_{2j})\hat{x}_j$ for all $F \in E$. Then $\sum y_i\omega^i \in I$ if and only if $\sum y_i\hat{\omega}^i(F)$ converges weakly in $\ell^2_{\mathbb{R}}(\mathbb{Z})$.

$$\sum y_i\hat{\omega}^i(F) = \sum_{i\in\mathbb{Z}} y_i \sum_{j\in\mathbb{Z}} (1+j^2)^{-\frac{3}{2}}\omega^i_j F(\lambda_{2j}) = \sum_{i\in\mathbb{Z}} y_i\pi_i G(\mu_i)$$

where $G \in E$ with $G(\lambda_{2j}) = (1+j^2)^{-3/2}F(\lambda_{2j})$ in using the interpolation theorem. Moreover, using equality (2.4), $A[G, 1_\alpha] = B[G, 1_\alpha] = G(\alpha)$ successively with $\alpha = \mu_i$ and $\alpha = \lambda_{2i}$, we can see that $G(\mu_i)$ is comparable to $G(\lambda_{2i})$ with constants independent of G and i. Consequently $\sum y_i \hat{\omega}^i(F)$ is comparable to $\sum y_i \pi_i (1+i^2)^{-\frac{3}{2}} F(\lambda_{2i})$, which converges if and only if $y \in I$.

(ii) From (i) if $x \in I/L$ then x can be written $\sum y_i \omega^i$ with $y_i \in [0,1[$. We obtain as in (i) and with the same notations, that for all $F \in E$, $\hat{x}(F)$ is comparable to $\sum y_i (1+j^2)^{-3/2}\pi_i F(\lambda_{2i})$. We then deduce that $|\hat{x}(F)| \leq C \sum |F(\lambda_{2j})|^2$ with a constant C independent of F. Since $\sum |F(\lambda_{2j})|^2$ is comparable to $\int_{-\infty}^{+\infty} |F(z)|^2 dz$ from $A[F,F] = B[F,F]$, we obtain

$$\|x\|_I = \|\hat{x}\|^2_{\ell^2_{\mathbb{C}}(\mathbb{Z})} = \sup_{\|F\|^2_E = 1} \|\hat{x}(F)\| \leq C$$

where the constant C is independent of $(y_i)_{i \in \mathbb{Z}}$. Proceeding exactly in the same way we obtain that $\lim_{j \to +\infty} \sum_{|i|j} |x_i|^2 (1+i^2)^{-3} = 0$ uniformly in $x \in I/L$. ∎

THEOREM 3.2 (Almost periodicity of NLS): *For any $\varepsilon > 0$ there exists $\ell(\varepsilon) < +\infty$ such that all intervals of length at least $\ell(\varepsilon)$ contain a real number $T > 0$ satisfying: the solution to* (1.1) *verifies for all $t \in \mathbb{R}$,*

$$\|u(\cdot, t+T) - u(\cdot, t)\|_{L^2_{\mathbb{C}}([0,1])} < \varepsilon,$$

independently of any initial data in $\mathrm{Iso}(u_0)$.

Proof of Theorem 3.2: Identifying $u(x,t)$ with $p(x,t) + iq(x,t)$ where p and q are real valued, it is now as for the KdV equation a consequence of the inequality (see [Mck-Tru, Lemma 11.2] for an analogous calculus)

$$\left\| \begin{pmatrix} p(\cdot, t+T) \\ q(\cdot, t+T) \end{pmatrix} - \begin{pmatrix} p(\cdot, t) \\ q(\cdot, t) \end{pmatrix} \right\|_{L^2_{\mathbb{R}}([0,1])} \leq C \inf_{\theta \in L} \|T\tau - \theta\|_I$$

for all reals t and T. The constant C depends only on $\mathrm{Iso}(p_0, q_0)$ and then not on T. The proof is finished by recalling that any flow of the form $T \mapsto x_0 + Tx$ is almost periodic on a compact set. ∎

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